2.5 GENERAL MOMENTUM EQUATIONS

In the previous sections of this chapter, we determined velocity distributions for some simple flow systems by applying differential momentum balances. The balances for these systems served to illustrate the application of the principle of conservation of momentum. In general, when dealing with isothermal fluid systems which do not involve changes in compositions, we can solve problems by starting with general expressions. This method is better than developing formulations peculiar to the specific problem at hand. The general momentum balance is called the *equation of motion* or the *Navier-Stokes' equation*; in addition the *equation of continuity* is frequently used in conjunction with the equation of motion.

The continuity equation is developed simply by applying the law of conservation of mass to a small volume element within a flowing fluid. The principle of conservation of mass is quite simple to apply and we assume that the reader has used it in developing material balances. We develop the equation of motion by applying the law of conservation of momentum which, in its general form, is an extension of Eq. (2.1). With the aid of these two equations, we can mathematically describe the problems encountered in the previous section, as well as more complicated problems. However, as we shall see, these expressions are rather cumbersome, and exact solutions can be found only in very limited cases. Hence these equations are used primarily as starting points for solving problems. The equations of continuity and motion are simplified to fit the problem at hand. Although theoretically these equations are valid for both laminar and turbulent flows, in practice they are applied only to laminar flow.

2.5.1 Equation of continuity

Consider the stationary volume element within a fluid moving with a velocity having the components v_x, v_y, and v_z, as shown in Fig. 2.4. We begin with the basic representation of the conservation of mass:

$$\begin{pmatrix}\text{rate of mass}\\ \text{accumulation}\end{pmatrix} = \begin{pmatrix}\text{rate of}\\ \text{mass in}\end{pmatrix} - \begin{pmatrix}\text{rate of}\\ \text{mass out}\end{pmatrix}. \tag{2.35}$$

First, look at the faces perpendicular to the x-axis. The volume flow rate of fluid in across the face at x is simply the product of the velocity (x-component) and the cross-sectional area, yielding $\Delta y\, \Delta z v_x|_x$. The rate of mass in through the face at x is then $\Delta y\, \Delta z(\rho v_x)|_x$. Similarly, the rate of mass out through the face at $x + \Delta x$ is $\Delta y\, \Delta z(\rho v_x)|_{x+\Delta x}$. We may write analogous expressions for the other two pairs of faces, and then enter all the terms that account for the fluid entering and leaving the system into the mass balance, and leave the accumulation term to be developed.

The *accumulation* is the rate of change of mass within the control volume

$$\Delta x\, \Delta y\, \Delta z \frac{\partial \rho}{\partial t}.$$

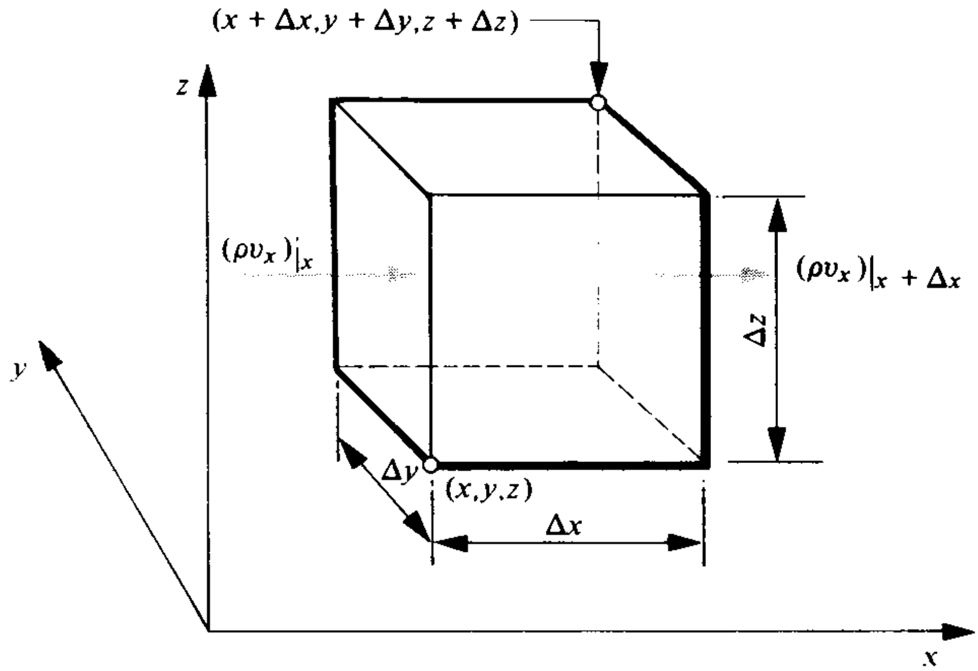

Fig. 2.4 Volume element fixed in space with fluid flowing through it.

The mass balance thus becomes

$$\Delta x\, \Delta y\, \Delta z \frac{\partial \rho}{\partial t} = \Delta y\, \Delta z[\rho v_x|_x - \rho v_x|_{x+\Delta x}]$$
$$+ \Delta x\, \Delta z[\rho v_y|_y - \rho v_y|_{y+\Delta y}]$$
$$+ \Delta x\, \Delta y[\rho v_z|_z - \rho v_z|_{z+\Delta z}]. \qquad (2.36)$$

Then, dividing through by $\Delta x\,\Delta y\,\Delta z$, and taking the limit as these dimensions approach zero, we get the *equation of continuity*:

$$\frac{\partial \rho}{\partial t} = -\left(\frac{\partial}{\partial x}\rho v_x + \frac{\partial}{\partial y}\rho v_y + \frac{\partial}{\partial z}\rho v_x\right). \tag{2.37}$$

A very important form of Eq. (2.37) is the form that applies to a fluid of constant density. For this case, which frequently occurs in engineering problems, the continuity equation reduces to

$$\frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} + \frac{\partial v_z}{\partial z} = 0. \tag{2.38}$$

2.5.2 Conservation of momentum

When Eq. (2.1) is extended to include unsteady-state systems, the momentum balance takes the form:

$$\begin{pmatrix}\text{rate of}\\ \text{momentum}\\ \text{accumulation}\end{pmatrix} = \begin{pmatrix}\text{rate of}\\ \text{momentum}\\ \text{in}\end{pmatrix} - \begin{pmatrix}\text{rate of}\\ \text{momentum}\\ \text{out}\end{pmatrix} + \begin{pmatrix}\text{sum of}\\ \text{forces acting}\\ \text{on system}\end{pmatrix}. \tag{2.39}$$

For simplicity, we begin by considering only the x-component of each term in Eq. (2.39); the y- and z-components may be handled in the same manner.

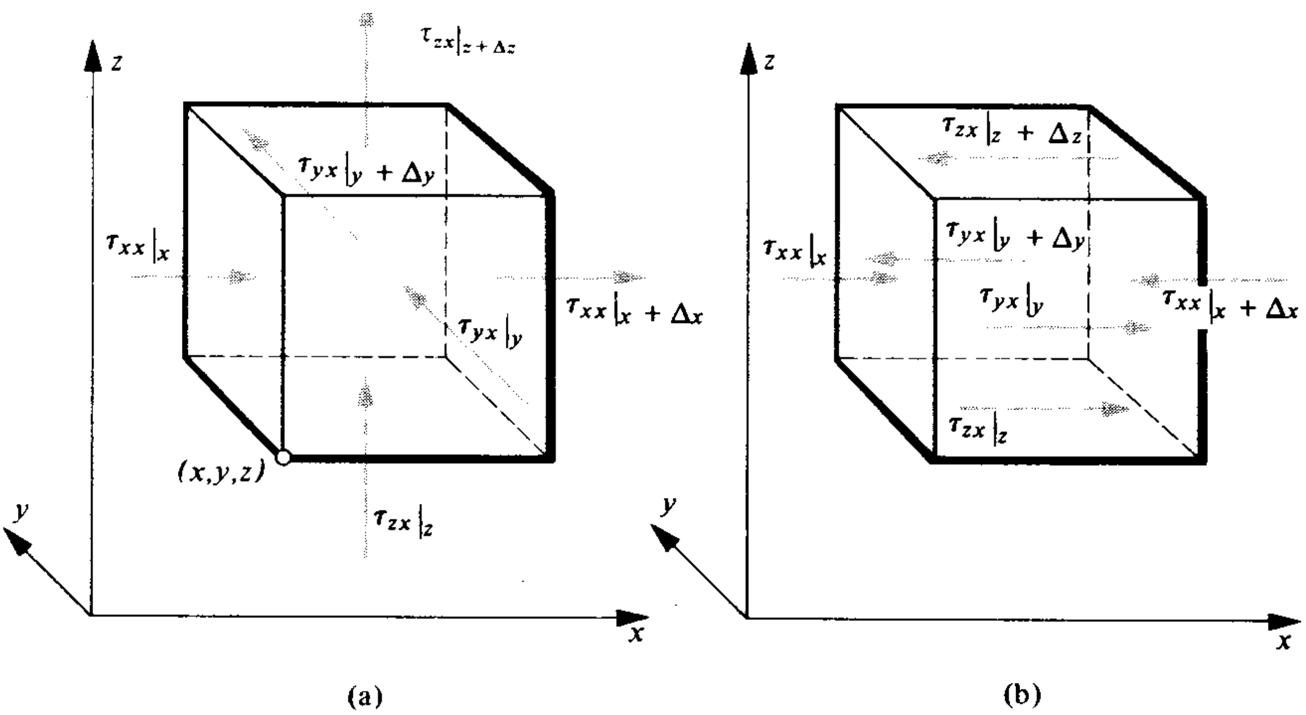

Fig. 2.5 Momentum transport (x-component) due to viscosity into the volume element. (a) Directions of viscous momentum transport. (b) Directions of forces.

Figure 2.5(a) shows the x-components of τ as if they were made up of viscous momentum fluxes rather than shear stresses. On the other hand Fig. 2.5(b) shows the x-components of τ as stresses. Note the appearance of τ_{xx}, which by the scheme of subscripts represents the transport of x-momentum in the x-direction. Alternatively, we view τ_{xx} as the x-directed *normal* stress on the x-face, in contrast to τ_{yx} which we view as the x-directed *shear* stress on the y-face.

Let us now develop the terms that enter into Eq. (2.39). First, the net rate at which the x-component of the *convective* momentum enters the unit volume, is

$$\Delta y\,\Delta z(\rho v_x v_x|_x - \rho v_x v_x|_{x+\Delta x}) + \Delta x\,\Delta z(\rho v_y v_x|_y - \rho v_y v_x|_{y+\Delta y}) + \Delta x\,\Delta y(\rho v_z v_x|_z - v_z v_x|_{z+\Delta z}). \quad (2.40)$$

Similarly, the net rate of *viscous* momentum flow into the unit volume across the six faces is

$$\Delta y\,\Delta z(\tau_{xx}|_x - \tau_{xx}|_{x+\Delta x}) + \Delta x\,\Delta z(\tau_{yx}|_y - \tau_{yx}|_{y+\Delta y}) + \Delta x\,\Delta y(\tau_{zx}|_z - \tau_{zx}|_{z+\Delta z}). \quad (2.41)$$

The reader who has not come in contact with this development before might find a brief explanation of the meaning of $\rho v_y v_x$ and $\rho v_z v_x$ useful. Remember that we are applying the law of conservation of momentum to the x-component of momentum. Thus v_x represents the x-velocity, and the rate at which mass enters the system through the y-face is given by $\Delta x\,\Delta z \rho v_y|_y$. Hence the rate at which x-momentum enters through the y-face is simply the product of mass-flow rate and velocity:

$$\Delta x\,\Delta z \rho v_y v_x|_y.$$

In most cases, the forces acting on the system are those arising from the pressure P and the gravitational force per unit mass g. In the x-direction, these forces are

$$\Delta y\, \Delta z(P|_x - P|_{x+\Delta x}), \tag{2.42}$$

and

$$\rho g_x\, \Delta x\, \Delta y\, \Delta z, \tag{2.43}$$

respectively. Here g_x is the x-component of the gravitational force. Finally, the rate of accumulation of x-momentum within the element is

$$\Delta x\, \Delta y\, \Delta z\left(\frac{\partial}{\partial t}\, \rho v_x\right). \tag{2.44}$$

Entering Eqs. (2.40)–(2.44) into the momentum balance, dividing through by $\Delta x\, \Delta y\, \Delta z$, and taking the limit as all three approach zero, we obtain the x-component of the momentum conservation equation:

$$\begin{aligned}\frac{\partial}{\partial t}\, \rho v_x = &-\left(\frac{\partial}{\partial x}\, \rho v_x v_x + \frac{\partial}{\partial y}\, \rho v_y v_x + \frac{\partial}{\partial z}\, \rho v_z v_x\right)\\ &-\left(\frac{\partial}{\partial x}\, \tau_{xx} + \frac{\partial}{\partial y}\, \tau_{yx} + \frac{\partial}{\partial z}\, \tau_{zx}\right)\\ &-\frac{\partial P}{\partial x} + \rho g_x.\end{aligned} \tag{2.45}$$

The y- and z-components, which we obtain in a similar manner, are

$$\frac{\partial}{\partial t}\rho v_y = -\left(\frac{\partial}{\partial x}\rho v_x v_y + \frac{\partial}{\partial y}\rho v_y v_y + \frac{\partial}{\partial z}\rho v_z v_y\right)$$
$$-\left(\frac{\partial}{\partial x}\tau_{xy} + \frac{\partial}{\partial y}\tau_{yy} + \frac{\partial}{\partial z}\tau_{zy}\right)$$
$$-\frac{\partial P}{\partial y} + \rho g_y, \tag{2.46}$$

and

$$\frac{\partial}{\partial t}\rho v_z = -\left(\frac{\partial}{\partial x}\rho v_x v_z + \frac{\partial}{\partial y}\rho v_y v_z + \frac{\partial}{\partial z}\rho v_z v_z\right)$$
$$-\left(\frac{\partial}{\partial x}\tau_{xz} + \frac{\partial}{\partial y}\tau_{yz} + \frac{\partial}{\partial z}\tau_{zz}\right) - \frac{\partial P}{\partial z} + g_z. \tag{2.47}$$

To describe the general case, all three Equations (2.45), (2.46), and (2.47 are needed. Vector notation can reduce these to one equation which is just as meaningful as all three. The quantities ρv_x, ρv_y, and ρv_z are the components of the mass velocity $\rho \boldsymbol{v}$; similarly g_x, g_y, and g_z are the components of $\boldsymbol{g}$. Vectorial representation of a velocity and an acceleration is familiar to most readers. However, the terms $\partial P/\partial x$, $\partial P/\partial y$, and $\partial P/\partial z$ all represent pressure *gradients*. By itself, pressure is a scalar quantity, but the gradient of pressure is a vector, denoted, in general by ∇P (sometimes written grad P).

As a simple example to illustrate the necessity of thinking about pressure gradients as vectors, take a tank of water. At any level L, measured from the surface of the water, the pressure is $\rho g L$. If the z-coordinate is that perpendicular to the surface of the water, then the pressure difference in the water *in the z-direction* is $\rho g L$. Note that we have to specify the *direction* in which the difference in pressure is measured; it would not be enough to say simply that the difference in pressure is $\rho g L$ because one could compare the pressure at two different points at the same level; hence the difference would be zero. To be specific, the pressure gradient at the level L must be denoted by $\partial P/\partial x = 0, \partial P/\partial y = 0$, and $\partial P/\partial z = \rho g$; being more general, the pressure gradient is ∇P. Therefore

$$\nabla P = \frac{\partial}{\partial x} P + \frac{\partial}{\partial y} P + \frac{\partial}{\partial z} P,$$

and ∇ can be thought to be an operator, such that

$$\nabla = \frac{\partial}{\partial x} + \frac{\partial}{\partial y} + \frac{\partial}{\partial z}.$$

The terms $\rho v_x v_x$, $\rho v_x v_y$, $\rho v_x v_z$, $\rho v_y v_z$, etc., are the nine components of the convective momentum flux $\rho \boldsymbol{vv}$, which is the *dyadic product* of $\rho \boldsymbol{v}$ and $\boldsymbol{v}$. Also τ_{xx}, τ_{xy}, etc., are the nine components of $\boldsymbol{\tau}$.

The vector equation representing Eqs. (2.45)–(2.47) is finally written:

$$\frac{\partial}{\partial t}\rho \boldsymbol{v} = -[\nabla \cdot \rho \boldsymbol{vv}] - \nabla P - [\nabla \cdot \boldsymbol{\tau}] + \rho \boldsymbol{g}. \tag{2.48}$$

Note here that ∇P is the product of a vector (∇) and a scalar (P), yielding a vector. To interpret the mathematical nature of $\nabla \cdot \rho \boldsymbol{vv}$ and $\nabla \cdot \boldsymbol{\tau}$ in physical terms is more difficult. However, for sufficient understanding of this text it is enough if the reader accepts them as mathematical shorthands of the appropriate terms in Eqs. (2.45)–(2.47).

So far we have developed a general expression, namely, Eq. (2.48), for the law of conservation of momentum. However, in order to use this equation for the determination of velocity distributions, it is necessary to insert expressions for the various stresses in terms of velocity gradients and fluid properties. The following equations are presented without proof because the arguments involved are quite lengthy. For *Newtonian fluids*, the nine components of $\boldsymbol{\tau}$ are written as follows.[1]

$$\text{Normal stresses}\begin{cases}\tau_{xx} = -2\eta\dfrac{\partial v_x}{\partial x} + \frac{2}{3}\eta(\nabla\cdot\boldsymbol{v}) & (2.49)\\[2ex] \tau_{yy} = -2\eta\dfrac{\partial v_y}{\partial y} + \frac{2}{3}\eta(\nabla\cdot\boldsymbol{v}) & (2.50)\\[2ex] \tau_{zz} = -2\eta\dfrac{\partial v_z}{\partial z} + \frac{2}{3}\eta(\nabla\cdot\boldsymbol{v}) & (2.51)\end{cases}$$

$$\text{Shear stresses}\begin{cases}\tau_{xy} = \tau_{yx} = -\eta\left(\dfrac{\partial v_x}{\partial y} + \dfrac{\partial v_y}{\partial x}\right) & (2.52)\\[2ex] \tau_{yz} = \tau_{zy} = -\eta\left(\dfrac{\partial v_y}{\partial z} + \dfrac{\partial v_z}{\partial y}\right) & (2.53)\\[2ex] \tau_{zx} = \tau_{xz} = -\eta\left(\dfrac{\partial v_z}{\partial x} + \dfrac{\partial v_x}{\partial z}\right) & (2.54)\end{cases}$$

These equations constitute a more general statement of Newton's law of viscosity than that given in Eq. (1.2), and apply to complex flow situations. When the fluid flows between two parallel plates in the x-direction so that v_x is a function of y alone, where the y-direction is perpendicular to the plates' surfaces (Fig. 1.4), then Eqs. (2.49)–(2.54) yield

$$\tau_{xx} = \tau_{yy} = \tau_{zz} = \tau_{yz} = \tau_{xz} = 0 \qquad \text{and} \qquad \tau_{yx} = -\eta(\partial v_x/\partial y),$$

which is the same as the simple relationship previously used to describe Newton's law of viscosity. Also in many other problems of physical significance in which v_x is recognized as a function of both y and x, we find that $\partial v_x/\partial y \gg \partial v_x/\partial x$, and the simple rate Eq. (1.2) can be used for τ_{yx} as an example with a high degree of accuracy rather than Eq. (2.52).

2.5.3 Navier-Stokes' equation, constant ρ and η

The continuity equation for constant density is given by Eq. (2.38) or in vector notation,

$$\nabla \cdot \boldsymbol{v} = 0. \tag{2.55}$$

Regarding the conservation of momentum, we can write Eqs. (2.45)–(2.47) with constant ρ and η:*

$$\rho\left[\frac{\partial v_x}{\partial t} + v_x\frac{\partial v_x}{\partial x} + v_y\frac{\partial v_x}{\partial y} + v_z\frac{\partial v_x}{\partial z}\right] = -\frac{\partial P}{\partial x} + \eta\left[\frac{\partial^2 v_x}{\partial x^2} + \frac{\partial^2 v_x}{\partial y^2} + \frac{\partial^2 v_x}{\partial z^2}\right] + \rho g_x, \tag{2.56}$$

$$\rho\left[\frac{\partial v_y}{\partial t} + v_y\frac{\partial v_y}{\partial y} + v_x\frac{\partial v_y}{\partial x} + v_z\frac{\partial v_y}{\partial z}\right] = -\frac{\partial P}{\partial y} + \eta\left[\frac{\partial^2 v_y}{\partial x^2} + \frac{\partial^2 v_y}{\partial y^2} + \frac{\partial^2 v_y}{\partial z^2}\right] + \rho g_y, \tag{2.57}$$

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* This development is the subject of Problem 2.7.

$$\rho\left[\frac{\partial v_z}{\partial t} + v_z\frac{\partial v_z}{\partial z} + v_x\frac{\partial v_z}{\partial x} + v_y\frac{\partial v_z}{\partial y}\right] = -\frac{\partial P}{\partial z} + \eta\left[\frac{\partial^2 v_z}{\partial x^2} + \frac{\partial^2 v_z}{\partial y^2} + \frac{\partial^2 v_z}{\partial z^2}\right] + \rho g_z. \quad (2.58)$$

The bracketed terms on the left side of these equations merit attention. Consider a control volume of fluid moving in space with no mass flow across its surface. The change in the x-component of its velocity with time and position is given by

$$\Delta v_x = \frac{\partial v_x}{\partial t}\Delta t + \frac{\partial v_x}{\partial x}\Delta x + \frac{\partial v_x}{\partial y}\Delta y + \frac{\partial v_x}{\partial z}\Delta z, \quad (2.59)$$

and since the x-component of acceleration is defined as

$$a_x = \lim_{\Delta t \to 0}\frac{\Delta v_x}{\Delta t} = \lim_{\Delta t \to 0}\left\{\frac{\partial v_x}{\partial t}\frac{\Delta t}{\Delta t} + \frac{\partial v_x}{\partial x}\frac{\Delta x}{\Delta t} + \frac{\partial v_x}{\partial y}\frac{\Delta y}{\Delta t} + \frac{\partial v_x}{\partial z}\frac{\Delta z}{\Delta t}\right\}, \quad (2.60)$$

we obtain

$$a_x = \frac{\partial v_x}{\partial t} + v_x\frac{\partial v_x}{\partial x} + v_y\frac{\partial v_x}{\partial y} + v_z\frac{\partial v_x}{\partial z} = \frac{Dv_x}{Dt}. \quad (2.61)$$

This is the acceleration one would feel if riding with the control volume of fluid. We also refer to this time derivative of velocity, Dv_x/Dt, as the *substantial* derivative. Analogous expressions exist for the y- and z-directions. In general, one notation can represent all three substantial derivatives, so that Eqs. (2.56)–(2.58) become

$$\rho \frac{D\boldsymbol{v}}{Dt} = -\nabla P + \eta \nabla^2 \boldsymbol{v} + \rho \boldsymbol{g}. \tag{2.62}$$

Equation (2.62), or Eqs. (2.56)–(2.58) which taken together represent the expansion of Eq. (2.62), is often referred to as the *Navier-Stokes' equation*. In the form of Eq. (2.62), we can recognize it as a statement of Newton's law in the form *mass* (ρ) $\times$ *acceleration* ($D\boldsymbol{v}/Dt$) equals the *sum of forces*, namely, the pressure force ($-\nabla P$), the viscous force ($\eta \nabla^2 \boldsymbol{v}$), and the gravity or body force $\rho \boldsymbol{g}$.

2.6 THE CONSERVATION OF MOMENTUM EQUATION IN CURVILINEAR COORDINATES

In many instances rectangular coordinates are not useful for analyzing problems. For example, in the Hagen-Poiseuille problem discussed in Section 2.4, we described the axial velocity v_z as a function of only a single variable r by employing cylindrical coordinates. If rectangular coordinates had been used instead, v_z would have been a very complicated function of x and y. Similarly, it would have been difficult to describe and apply the boundary condition at the tube wall.

The equations of continuity and motion in Section 2.5 have been given in rectangular coordinates; spherical or cylindrical coordinates are presented in Tables 2.1–2.7.

Table 2.1 The continuity equation in different coordinate systems*

Rectangular coordinates (x, y, z):

$$\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x}(\rho v_x) + \frac{\partial}{\partial y}(\rho v_y) + \frac{\partial}{\partial z}(\rho v_z) = 0 \tag{A}$$

Cylindrical coordinates (r, θ, z):

$$\frac{\partial \rho}{\partial t} + \frac{1}{r}\frac{\partial}{\partial r}(\rho r v_r) + \frac{1}{r}\frac{\partial}{\partial \theta}(\rho v_\theta) + \frac{\partial}{\partial z}(\rho v_z) = 0 \tag{B}$$

Spherical coordinates (r, θ, ϕ):

$$\frac{\partial \rho}{\partial t} + \frac{1}{r^2}\frac{\partial}{\partial r}(\rho r^2 v_r) + \frac{1}{r \sin\theta}\frac{\partial}{\partial \theta}(\rho v_\theta \sin\theta) + \frac{1}{r \sin\theta}\frac{\partial}{\partial \phi}(\rho v_\phi) = 0 \tag{C}$$

* Tables 2.1–2.7 are from R. B. Bird, W. E. Stewart, and E. N. Lightfoot, *Transport Phenomena*, Wiley, New York, 1960, pages 83–91. Reprinted by permission.

Table 2.2 The conservation of momentum in rectangular coordinates (x, y, z)

In terms of τ:

x-component
$$\rho\left(\frac{\partial v_x}{\partial t} + v_x\frac{\partial v_x}{\partial x} + v_y\frac{\partial v_x}{\partial y} + v_z\frac{\partial v_x}{\partial z}\right) = -\frac{\partial P}{\partial x} - \left(\frac{\partial \tau_{xx}}{\partial x} + \frac{\partial \tau_{yx}}{\partial y} + \frac{\partial \tau_{zx}}{\partial z}\right) + \rho g_x \qquad \text{(A)}$$

y-component
$$\rho\left(\frac{\partial v_y}{\partial t} + v_x\frac{\partial v_y}{\partial x} + v_y\frac{\partial v_y}{\partial y} + v_z\frac{\partial v_y}{\partial z}\right) = -\frac{\partial P}{\partial y} - \left(\frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \tau_{yy}}{\partial y} + \frac{\partial \tau_{zy}}{\partial z}\right) + \rho g_y \qquad \text{(B)}$$

z-component
$$\rho\left(\frac{\partial v_z}{\partial t} + v_x\frac{\partial v_z}{\partial x} + v_y\frac{\partial v_z}{\partial y} + v_z\frac{\partial v_z}{\partial z}\right) = -\frac{\partial P}{\partial z} - \left(\frac{\partial \tau_{xz}}{\partial x} + \frac{\partial \tau_{yz}}{\partial y} + \frac{\partial \tau_{zz}}{\partial z}\right) + \rho g_z \qquad \text{(C)}$$

In terms of velocity gradients for a Newtonian fluid with constant ρ and η:

x-component
$$\rho\left(\frac{\partial v_x}{\partial t} + v_x\frac{\partial v_x}{\partial x} + v_y\frac{\partial v_x}{\partial y} + v_z\frac{\partial v_x}{\partial z}\right) = -\frac{\partial P}{\partial x} + \eta\left(\frac{\partial^2 v_x}{\partial x^2} + \frac{\partial^2 v_x}{\partial y^2} + \frac{\partial^2 v_x}{\partial z^2}\right) + \rho g_x \qquad \text{(D)}$$

y-component
$$\rho\left(\frac{\partial v_y}{\partial t} + v_x\frac{\partial v_y}{\partial x} + v_y\frac{\partial v_y}{\partial y} + v_z\frac{\partial v_y}{\partial z}\right) = -\frac{\partial P}{\partial y} + \eta\left(\frac{\partial^2 v_y}{\partial x^2} + \frac{\partial^2 v_y}{\partial y^2} + \frac{\partial^2 v_y}{\partial z^2}\right) + \rho g_y \qquad \text{(E)}$$

z-component
$$\rho\left(\frac{\partial v_z}{\partial t} + v_x\frac{\partial v_z}{\partial x} + v_y\frac{\partial v_z}{\partial y} + v_z\frac{\partial v_z}{\partial z}\right) = -\frac{\partial P}{\partial z} + \eta\left(\frac{\partial^2 v_z}{\partial x^2} + \frac{\partial^2 v_z}{\partial y^2} + \frac{\partial^2 v_z}{\partial z^2}\right) + \rho g_z \qquad \text{(F)}$$

Table 2.3 The conservation of momentum in cylindrical coordinates (r, θ, z)

In terms of τ:

r-component*
$$\rho\left(\frac{\partial v_r}{\partial t} + v_r \frac{\partial v_r}{\partial r} + \frac{v_\theta}{r}\frac{\partial v_r}{\partial \theta} - \frac{v_\theta^2}{r} + v_z \frac{\partial v_r}{\partial z}\right) = -\frac{\partial P}{\partial r}$$
$$-\left(\frac{1}{r}\frac{\partial}{\partial r}(r\tau_{rr}) + \frac{1}{r}\frac{\partial \tau_{r\theta}}{\partial \theta} - \frac{\tau_{\theta\theta}}{r} + \frac{\partial \tau_{rz}}{\partial z}\right) + \rho g_r \quad \text{(A)}$$

θ-component
$$\rho\left(\frac{\partial v_\theta}{\partial t} + v_r \frac{\partial v_\theta}{\partial r} + \frac{v_\theta}{r}\frac{\partial v_\theta}{\partial \theta} + \frac{v_r v_\theta}{r} + v_z \frac{\partial v_\theta}{\partial z}\right) = -\frac{1}{r}\frac{\partial P}{\partial \theta}$$
$$-\left(\frac{1}{r^2}\frac{\partial}{\partial r}(r^2\tau_{r\theta}) + \frac{1}{r}\frac{\partial \tau_{\theta\theta}}{\partial \theta} + \frac{\partial \tau_{\theta z}}{\partial z}\right) + \rho g_\theta \quad \text{(B)}$$

z-component
$$\rho\left(\frac{\partial v_z}{\partial t} + v_r \frac{\partial v_z}{\partial r} + \frac{v_\theta}{r}\frac{\partial v_z}{\partial \theta} + v_z \frac{\partial v_z}{\partial z}\right) = -\frac{\partial P}{\partial z}$$
$$-\left(\frac{1}{r}\frac{\partial}{\partial r}(r\tau_{rz}) + \frac{1}{r}\frac{\partial \tau_{\theta z}}{\partial \theta} + \frac{\partial \tau_{zz}}{\partial z}\right) + \rho g_z \quad \text{(C)}$$

In terms of velocity gradients for a Newtonian fluid with constant ρ and η:

r-component*
$$\rho\left(\frac{\partial v_r}{\partial t} + v_r \frac{\partial v_r}{\partial r} + \frac{v_\theta}{r}\frac{\partial v_r}{\partial \theta} - \frac{v_\theta^2}{r} + v_z \frac{\partial v_r}{\partial z}\right) = -\frac{\partial P}{\partial r}$$
$$+\eta\left[\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial}{\partial r}(r v_r)\right) + \frac{1}{r^2}\frac{\partial^2 v_r}{\partial \theta^2} - \frac{2}{r^2}\frac{\partial v_\theta}{\partial \theta} + \frac{\partial^2 v_r}{\partial z^2}\right] + \rho g_r \quad \text{(D)}$$

θ-component
$$\rho\left(\frac{\partial v_\theta}{\partial t} + v_r \frac{\partial v_\theta}{\partial r} + \frac{v_\theta}{r}\frac{\partial v_\theta}{\partial \theta} + \frac{v_r v_\theta}{r} + v_z \frac{\partial v_\theta}{\partial z}\right) = -\frac{1}{r}\frac{\partial P}{\partial \theta}$$
$$+\eta\left[\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial}{\partial r}(r v_\theta)\right) + \frac{1}{r^2}\frac{\partial^2 v_\theta}{\partial \theta^2} + \frac{2}{r^2}\frac{\partial v_r}{\partial \theta} + \frac{\partial^2 v_\theta}{\partial z^2}\right] + \rho g_\theta \quad \text{(E)}$$

z-component
$$\rho\left(\frac{\partial v_z}{\partial t} + v_r \frac{\partial v_z}{\partial r} + \frac{v_\theta}{r}\frac{\partial v_z}{\partial \theta} + v_z \frac{\partial v_z}{\partial z}\right) = -\frac{\partial P}{\partial z}$$
$$+\eta\left[\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial v_z}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 v_z}{\partial \theta^2} + \frac{\partial^2 v_z}{\partial z^2}\right] + \rho g_z \quad \text{(F)}$$

* The term $\rho v_\theta^2/r$ is the *centrifugal force*. It gives the effective force in the r-direction resulting from fluid motion in the θ-direction. This term arises automatically on transformation from rectangular to cylindrical coordinates; it does not have to be added on physical grounds.

Table 2.4 The conservation of momentum in spherical coordinates (r, θ, ϕ)

In terms of τ:

r-component

$$\rho\left(\frac{\partial v_r}{\partial t} + v_r \frac{\partial v_r}{\partial r} + \frac{v_\theta}{r}\frac{\partial v_r}{\partial \theta} + \frac{v_\phi}{r \sin\theta}\frac{\partial v_r}{\partial \phi} - \frac{v_\theta^2 + v_\phi^2}{r}\right)$$

$$= -\frac{\partial P}{\partial r} - \left(\frac{1}{r^2}\frac{\partial}{\partial r}(r^2 \tau_{rr}) + \frac{1}{r \sin\theta}\frac{\partial}{\partial \theta}(\tau_{r\theta}\sin\theta) + \frac{1}{r \sin\theta}\frac{\partial \tau_{r\phi}}{\partial \phi} - \frac{\tau_{\theta\theta} + \tau_{\phi\phi}}{r}\right) + \rho g_r \qquad \text{(A)}$$

θ-component

$$\rho\left(\frac{\partial v_\theta}{\partial t} + v_r \frac{\partial v_\theta}{\partial r} + \frac{v_\theta}{r}\frac{\partial v_\theta}{\partial \theta} + \frac{v_\phi}{r \sin\theta}\frac{\partial v_\theta}{\partial \phi} + \frac{v_r v_\theta}{r} - \frac{v_\phi^2 \cot\theta}{r}\right)$$

$$= -\frac{1}{r}\frac{\partial P}{\partial \theta} - \left(\frac{1}{r^2}\frac{\partial}{\partial r}(r^2 \tau_{r\theta}) + \frac{1}{r \sin\theta}\frac{\partial}{\partial \theta}(\tau_{\theta\theta}\sin\theta) + \frac{1}{r \sin\theta}\frac{\partial \tau_{\theta\phi}}{\partial \phi} + \frac{\tau_{r\theta}}{r} - \frac{\cot\theta}{r}\tau_{\phi\phi}\right) + \rho g_\theta \qquad \text{(B)}$$

ϕ-component

$$\rho\left(\frac{\partial v_\phi}{\partial t} + v_r \frac{\partial v_\phi}{\partial r} + \frac{v_\theta}{r}\frac{\partial v_\phi}{\partial \theta} + \frac{v_\phi}{r \sin\theta}\frac{\partial v_\phi}{\partial \phi} + \frac{v_\phi v_r}{r} + \frac{v_\theta v_\phi}{r}\cot\theta\right)$$

$$= -\frac{1}{r \sin\theta}\frac{\partial P}{\partial \phi} - \left(\frac{1}{r^2}\frac{\partial}{\partial r}(r^2 \tau_{r\phi}) + \frac{1}{r}\frac{\partial \tau_{\theta\phi}}{\partial \theta} + \frac{1}{r \sin\theta}\frac{\partial \tau_{\phi\phi}}{\partial \phi} + \frac{\tau_{r\phi}}{r} + \frac{2\cot\theta}{r}\tau_{\theta\phi}\right) + \rho g_\phi \qquad \text{(C)}$$

In terms of velocity gradients for a Newtonian fluid with constant ρ and η:

r-component

$$\rho\left(\frac{\partial v_r}{\partial t} + v_r \frac{\partial v_r}{\partial r} + \frac{v_\theta}{r}\frac{\partial v_r}{\partial \theta} + \frac{v_\phi}{r \sin\theta}\frac{\partial v_r}{\partial \phi} - \frac{v_\theta^2 + v_\phi^2}{r}\right)$$

$$= -\frac{\partial P}{\partial r} + \eta\left(\nabla^2 v_r - \frac{2}{r^2}v_r - \frac{2}{r^2}\frac{\partial v_\theta}{\partial \theta} - \frac{2}{r^2}v_\theta\cot\theta - \frac{2}{r^2 \sin\theta}\frac{\partial v_\phi}{\partial \phi}\right) + \rho g_r \qquad \text{(D)}$$

θ-component

$$\rho\left(\frac{\partial v_\theta}{\partial t} + v_r \frac{\partial v_\theta}{\partial r} + \frac{v_\theta}{r}\frac{\partial v_\theta}{\partial \theta} + \frac{v_\phi}{r \sin\theta}\frac{\partial v_\theta}{\partial \phi} + \frac{v_r v_\theta}{r} - \frac{v_\phi^2 \cot\theta}{r}\right)$$

$$= -\frac{1}{r}\frac{\partial P}{\partial \theta} + \eta\left(\nabla^2 v_\theta + \frac{2}{r^2}\frac{\partial v_r}{\partial \theta} - \frac{v_\theta}{r^2 \sin^2\theta} - \frac{2\cos\theta}{r^2 \sin^2\theta}\frac{\partial v_\phi}{\partial \phi}\right) + \rho g_\theta \qquad \text{(E)}$$

ϕ-component

$$\rho\left(\frac{\partial v_\phi}{\partial t} + v_r \frac{\partial v_\phi}{\partial r} + \frac{v_\theta}{r}\frac{\partial v_\phi}{\partial \theta} + \frac{v_\phi}{r \sin\theta}\frac{\partial v_\phi}{\partial \phi} + \frac{v_\phi v_r}{r} + \frac{v_\theta v_\phi}{r}\cot\theta\right)$$

$$= -\frac{1}{r \sin\theta}\frac{\partial P}{\partial \phi} + \eta\left(\nabla^2 v_\phi - \frac{v_\phi}{r^2 \sin^2\theta} + \frac{2}{r^2 \sin\theta}\frac{\partial v_r}{\partial \phi} + \frac{2\cos\theta}{r^2 \sin^2\theta}\frac{\partial v_\theta}{\partial \phi}\right) + \rho g_\phi \qquad \text{(F)}$$

Table 2.5 Components of the stress tensor in rectangular coordinates (x, y, z)

$$\tau_{xx} = -\eta\left[2\frac{\partial v_x}{\partial x} - \tfrac{2}{3}(\nabla \cdot \boldsymbol{v})\right] \tag{A}$$

$$\tau_{yy} = -\eta\left[2\frac{\partial v_y}{\partial y} - \tfrac{2}{3}(\nabla \cdot \boldsymbol{v})\right] \tag{B}$$

$$\tau_{zz} = -\eta\left[2\frac{\partial v_z}{\partial z} - \tfrac{2}{3}(\nabla \cdot \boldsymbol{v})\right] \tag{C}$$

$$\tau_{xy} = \tau_{yx} = -\eta\left[\frac{\partial v_x}{\partial y} + \frac{\partial v_y}{\partial x}\right] \tag{D}$$

$$\tau_{yz} = \tau_{zy} = -\eta\left[\frac{\partial v_y}{\partial z} + \frac{\partial v_z}{\partial y}\right] \tag{E}$$

$$\tau_{zx} = \tau_{xz} = -\eta\left[\frac{\partial v_z}{\partial x} + \frac{\partial v_x}{\partial z}\right] \tag{F}$$

$$(\nabla \cdot \boldsymbol{v}) = \frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} + \frac{\partial v_z}{\partial z} \tag{G}$$

Table 2.6 Components of the stress tensor in cylindrical coordinates (r, θ, z)

$$\tau_{rr} = -\eta\left[2\frac{\partial v_r}{\partial r} - \tfrac{2}{3}(\nabla \cdot \boldsymbol{v})\right] \tag{A}$$

$$\tau_{\theta\theta} = -\eta\left[2\left(\frac{1}{r}\frac{\partial v_\theta}{\partial \theta} + \frac{v_r}{r}\right) - \tfrac{2}{3}(\nabla \cdot \boldsymbol{v})\right] \tag{B}$$

$$\tau_{zz} = -\eta\left[2\frac{\partial v_z}{\partial z} - \tfrac{2}{3}(\nabla \cdot \boldsymbol{v})\right] \tag{C}$$

$$\tau_{r\theta} = \tau_{\theta r} = -\eta\left[r\frac{\partial}{\partial r}\left(\frac{v_\theta}{r}\right) + \frac{1}{r}\frac{\partial v_r}{\partial \theta}\right] \tag{D}$$

$$\tau_{\theta z} = \tau_{z\theta} = -\eta\left[\frac{\partial v_\theta}{\partial z} + \frac{1}{r}\frac{\partial v_z}{\partial \theta}\right] \tag{E}$$

$$\tau_{zr} = \tau_{rz} = -\eta\left[\frac{\partial v_z}{\partial r} + \frac{\partial v_r}{\partial z}\right] \tag{F}$$

$$(\nabla \cdot \boldsymbol{v}) = \frac{1}{r}\frac{\partial}{\partial r}(r v_r) + \frac{1}{r}\frac{\partial v_\theta}{\partial \theta} + \frac{\partial v_z}{\partial z} \tag{G}$$